

Krylov-Veretennikov Formula for Functionals from the Stopped Wiener Process

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Abstract

We consider a class of measures absolutely continuous with respect to the distribution of the stopped Wiener process $w(\cdot \wedge \tau)$. Multiple stochastic integrals, that lead to the analogue of the Itô-Wiener expansions for such measures, are described. An analogue of the Krylov-Veretennikov formula for functionals $f = \varphi(w(\tau))$ is obtained.

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1 Introduction

Let $\{w(t)\}_{t \geq 0}$ be a standard Wiener process in \mathbb{R}^d , starting from the point $u \in \mathbb{R}^d$. Consider an open connected set $G \ni u$, the exit time

$$\tau = \inf\{t > 0 : w(t) \notin G\},$$

and a Borel function $\rho : \mathbb{R}^d \rightarrow (0, 1)$.

The main object of the investigation in the present paper is the orthogonal structure of the space $L^2(\Omega, \sigma(w(\cdot \wedge \tau)), Q)$, where the measure Q is given by the density

$$\frac{dQ}{d\mathbb{P}} = \frac{1_{\tau < \infty} \rho(w(\tau))}{\mathbb{E} 1_{\tau < \infty} \rho(w(\tau))}.$$

In [11, L. 2.4] it was proved that the space $L^2(\Omega, \sigma(w(\cdot \wedge \tau)), Q)$ possesses an orthogonal structure similar to the Itô-Wiener decomposition in the Gaussian case [1, 5, 8]. Namely, consider functions

$$\beta(v) = \mathbb{E} 1_{\tau(w-u+v) < \infty} \rho(w(\tau(w-u+v) - u + v)), \quad v \in G,$$

$$\alpha(s, v) = \beta^{-1}(v) \mathbb{E} 1_{s < \tau(w-u+v) < \infty} \rho(w(\tau(w-u+v) - u + v)), \quad s > 0, v \in G,$$

and processes

$$\begin{aligned} \tilde{w}(s) &= w(s \wedge \tau) - \int_0^{s \wedge \tau} \nabla \log \beta(w(r)) dr, \quad s \geq 0; \\ \hat{w}_t(s) &= \tilde{w}(s) - \int_0^{s \wedge \tau} \nabla \log \alpha(t-r, w(r)) dr, \quad 0 \leq s \leq t. \end{aligned}$$

Theorem 1.1. [11, L. 2.4] *Each random variable $f \in L^2(\Omega, \sigma(w(\cdot \wedge \tau)), Q)$ can be uniquely represented as a series of pairwise orthogonal stochastic integrals*

$$f = \sum_{n=0}^{\infty} \int \dots \int_{0 < t_1 < \dots < t_n < \tau} a_n(t_1, \dots, t_n) d\hat{w}_{t_n}(t_1) \dots d\hat{w}_{t_n}(t_{n-1}) d\tilde{w}(t_n). \quad (1.1)$$

Conversely, given a sequence of Borel functions $a_n : (0, \infty)^n \rightarrow \mathbb{R}^n$, $n \geq 0$, such that

$$\sum_{n=0}^{\infty} \int \dots \int_{0 < t_1 < \dots < t_n} \alpha(t_n, u) |a_n(t_1, \dots, t_n)|^2 dt_1 \dots dt_n < \infty,$$

the series in the right-hand side of (1.1) converges in $L^2(\Omega, \sigma(w(\cdot \wedge \tau)), Q)$, and its sum f satisfies

$$\mathbb{E}f^2 = \sum_{n=0}^{\infty} \int \dots \int_{0 < t_1 < \dots < t_n} \alpha(t_n, u) |a_n(t_1, \dots, t_n)|^2 dt_1 \dots dt_n.$$

In this paper we derive the explicit form of the expansion (1.1) for random variables of the kind $f = \varphi(w(\tau))$. The resulting formula is similar to the well-known Krylov-Veretennikov formula [6]. It is written in terms of the transition semigroup $\{T_t^k\}_{t \geq 0}$ of a certain diffusion, killed at the boundary of G . Indeed, the process \tilde{w} is a stopped Wiener process relatively to the measure Q [11, L. 2.4]. Respectively, the initial Wiener process w is a diffusion process relatively to the measure Q . Then $\{T_t^k\}_{t \geq 0}$ is the transition semigroup of the process w killed at the boundary of G . Let T denote the integration with respect to the exit distribution of w from G (precise expressions for these operators are given in the section 2). The main result of the present paper is the following formula, proved in the theorem 2.1:

for every random variable $\varphi(w(\tau)) \in L^2(\Omega, \sigma(w(\cdot \wedge \tau)), Q)$ the expansion (1.1) has the form

$$\begin{aligned} \varphi(w(\tau)) = \sum_{n=0}^{\infty} \int \dots \int_{0 < t_1 < \dots < t_n} \alpha(t_n, u)^{-1} & \left(T_{t_1}^k \alpha(t_2 - t_1, \cdot) \nabla (\alpha(t_2 - t_1, \cdot)^{-1} T_{t_2 - t_1}^k) \dots \right. \\ & \left. \alpha(t_n - t_{n-1}, \cdot) \nabla (\alpha(t_n - t_{n-1}, \cdot)^{-1} T_{t_n - t_{n-1}}^k) \nabla T \varphi \right) (u) d\hat{w}_{t_n}(t_1) \dots d\hat{w}_{t_n}(t_{n-1}) d\tilde{w}(t_n). \end{aligned} \quad (1.2)$$

Expansions of the kind (1.1) appeared in [4] in connection with the problem of studying the behaviour of Gaussian measures under nonlinear transformations. Such expansions have two main features:

1. the summands in (1.1) are pairwise orthogonal;
2. the summands in (1.1) are $\sigma(w(\cdot \wedge \tau))$ -measurable.

Of course, there are other possibilities to organize series expansions for random variables from $L^2(\Omega, \sigma(w(\cdot \wedge \tau)), Q)$. For simplicity, consider the case $Q = \mathbb{P}$. The most straightforward approach comes from the obvious inclusion $\sigma(w(\cdot \wedge \tau)) \subset \sigma(w)$. It means that each

random variable $f \in L^2(\Omega, \sigma(w(\cdot \wedge \tau)), \mathbb{P})$ possesses an Itô-Wiener expansion with respect to the Wiener process w :

$$f = \sum_{n=0}^{\infty} \int \dots \int_{0 < t_1 < \dots < t_n} b_n(t_1, \dots, t_n) dw(t_1) \dots dw(t_n). \quad (1.3)$$

The summands in the expansion are not $\sigma(w(\cdot \wedge \tau))$ -measurable. While the left-hand side of (1.3) is $\sigma(w(\cdot \wedge \tau))$ -measurable, one can condition (1.3) with respect to $w(\cdot \wedge \tau)$ and get another expansion

$$f = \sum_{n=0}^{\infty} \int \dots \int_{0 < t_1 < \dots < t_n < \tau} b_n(t_1, \dots, t_n) dw(t_1) \dots dw(t_n). \quad (1.4)$$

Now the stochastic integrals of different degree are not orthogonal. This causes known inconveniences: the expansion (1.4) is not unique (an example is given in [4]); the conditions for the expression in the right-hand side of (1.4) to converge are complicated. An application of the Gram-Shmidt orthogonalization procedure to expansions (1.4) was considered in [2]. However, in our framework it seems to be too complicated either to obtain the orthogonalized form of (1.4), or to find the orthogonalized expansion (1.4) for a concrete random variable f . The expansion (1.1) overcomes all these problems.

Motivation for the $\sigma(w(\cdot \wedge \tau))$ -measurability of the summands in (1.1) comes from B. S. Tsirelson's theory of black noise. It is well-known that Brownian coalescing flows produce filtrations with trivial Gaussian parts [12, 7]. So, to get a unified description of functionals measurable with respect to such flows, it is reasonable to use the noise generated by the flow itself. The results from [4, 11] show that this idea works: in [4] an orthogonal expansion of the kind (1.1) was obtained for the stopped Brownian motion; in [11] the same was done for the n -point motions of the Arratia flow. We refer to [4, 11] for the detailed discussion of this and related questions.

Generalization of the Krylov-Veretennikov formula to the wide class of dynamical systems driven by the additive Gaussian noise was obtained in [3]. Our formula (2.7) is similar to the one obtained in [3] despite the additional multipliers α . They occur to normalize operators T_t^k , as $T_t^k 1 = \alpha(t, \cdot)$.

The article is organized in the following way. In the section 2 we introduce all the needed notions and constructions. Also, it contains the reduction of the main theorem 2.1 to lemmata 2.1 and 2.2. Sections 3 and 4 are devoted to the proof of these auxiliary results.

2 Notations and Main Results

To formulate our results, we will use the following notations.

$\{w(t)\}_{t \geq 0}$ is the Wiener process in \mathbb{R}^d . Without loss of generality, we will assume that w is constructed in a canonical way:

$\Omega = C([0, \infty), \mathbb{R}^d)$ is a space of continuous functions equipped with a metric of uniform convergence on compacts;

\mathcal{F} is the Borel σ -field on Ω ;

$w(t, \omega) = \omega(t)$ is the canonical process on (Ω, \mathcal{F}) , $\mathcal{F}_t = \sigma(w(s) : 0 \leq s \leq t)$ is the natural filtration of w ;

$(\mathbb{P}_v)_{v \in \mathbb{R}^d}$ is a family of probability measures on (Ω, \mathcal{F}) , such that relatively to \mathbb{P}_v , w is a d -dimensional Wiener process starting from v . The expectation with respect to certain probability measure Q on (Ω, \mathcal{F}) will be denoted by \mathbb{E}_Q . $\mathbb{E}_{\mathbb{P}_v}$ will be abbreviated to \mathbb{E}_v .

Let $G \subset \mathbb{R}^d$ be an open connected set, τ be the exit time of w from the set G :

$$\tau = \inf\{t > 0 : w(t) \notin G\}.$$

We will assume that for all $v \in G$, $\mathbb{P}_v(\tau < \infty) > 0$. Fix a Borel function $\rho : \mathbb{R}^d \rightarrow (0, 1)$ and consider the function

$$\beta(v) = \mathbb{E}_v 1_{\tau < \infty} \rho(w(\tau)), \quad v \in G.$$

It is a harmonic function in G [9, Ch. 4, Prop. 2.1]:

$$\Delta_v \beta(v) = 0, \quad v \in G.$$

Denote Q_u the probability measure on (Ω, \mathcal{F}) , defined via the density

$$\frac{dQ_u}{d\mathbb{P}_u} = \beta(u)^{-1} 1_{\tau < \infty} \rho(w(\tau)).$$

We will need another probability measure corresponding to the process w killed at the moment τ . Consider the function

$$\alpha(s, v) = Q_v(\tau > s), \quad s > 0, v \in G.$$

In the section 1 following processes were introduced.

$$\tilde{w}(s) = w(s \wedge \tau) - \int_0^{s \wedge \tau} \nabla_v \log \beta(w(r)) dr, \quad s \geq 0; \quad (2.5)$$

$$\hat{w}_t(s) = \tilde{w}(s) - \int_0^{s \wedge \tau} \nabla_v \log \alpha(t - r, w(r)) dr, \quad 0 \leq s \leq t. \quad (2.6)$$

Throughout the paper derivatives will be taken in $v \in G$, so we will omit the index v in the derivatives' notation.

Consider a probability measure $Q_{t,u}$ on (Ω, \mathcal{F}_t) , defined via the density

$$\frac{dQ_{t,u}}{dQ_u} = \alpha(t, u)^{-1} 1_{\tau > t}.$$

The key observation leading to the theorem 1.1 is that on the probability space $(\Omega, \mathcal{F}_t, Q_{t,u})$ the process \hat{w}_t is a Wiener process [10, Ch. VIII, Th. (1.4)].

Introduce following operators:

1. $T\psi(v) = \mathbb{E}_v 1_{\tau < \infty} \rho(w(\tau)) \psi(w(\tau)), v \in G.$

Denote μ_v the distribution of $w(\tau)$ relatively to the measure $1_{\tau < \infty} d\mathbb{P}_v$. Then the action of the operator T reduces to the integration with respect to μ_v :

$$T\psi(v) = \int \psi(x) \mu_v(dx).$$

2. $\tilde{T}\psi(v) = \beta(v)^{-1} T\psi(v), v \in G.$

The operator \tilde{T} is the expectation relatively to the probability measure Q_v :

$$\tilde{T}\psi(v) = \mathbb{E}_{Q_v} \rho(w(\tau)).$$

3. $T_s^k \psi(v) = \mathbb{E}_{Q_v} 1_{\tau > s} \psi(w(s)), s > 0, v \in G.$

From equations (2.5), (2.6) it follows that

$$dw(s) = (\nabla \log \alpha(t-s, w(s)) + \nabla \log \beta(w(s))) ds + d\hat{w}(s),$$

where \hat{w} is a Wiener process on $(\Omega, \mathcal{F}_t, Q_{t,u})$. So, relatively to the measure $Q_{t,u}$ the process w satisfies (degenerate) SDE. Respectively, $\{T_t^k\}_{t \geq 0}$ is the transition semigroup of a killed diffusion process w . Denote $\mu_{s,v}$ the distribution of $w(s)$ relatively to the measure $1_{\tau > s} dQ_v$. Then the action of the operator T_s^k reduces to the integration with respect to $\mu_{s,v}$:

$$T_s^k \psi(v) = \int \psi(x) \mu_{s,v}(dx).$$

4. $\widetilde{T}_s^k \psi(v) = \alpha(s, v)^{-1} T_s^k \psi(v), s > 0, v \in G.$

The operator \widetilde{T}_s^k is the expectation relatively to the probability measure $Q_{s,v}$:

$$\widetilde{T}_s^k \psi(v) = \mathbb{E}_{Q_{s,v}} \psi(w(s)).$$

The following theorem is the main result of the paper.

Theorem 2.1. *For every $\varphi \in L^2(\rho d\mu_u)$ the expansion (1.1) has the form*

$$\begin{aligned} \varphi(w(\tau)) = \sum_{n=0}^{\infty} \int \dots \int \alpha(t_n, u)^{-1} & \left(\alpha(t_1, \cdot) \widetilde{T}_{t_1}^k \alpha(t_2 - t_1, \cdot) \nabla \widetilde{T}_{t_2 - t_1}^k \dots \right. \\ & \left. \alpha(t_n - t_{n-1}, \cdot) \nabla \widetilde{T}_{t_n - t_{n-1}}^k \nabla \widetilde{T} \varphi \right)(u) d\hat{w}_{t_n}(t_1) \dots d\hat{w}_{t_n}(t_{n-1}) d\tilde{w}(t_n). \end{aligned} \quad (2.7)$$

The proof is divided into two lemmas, which are proved in the next sections. At first we derive the Clark representation for $\varphi(w(\tau))$ with respect to the stopped Wiener process \tilde{w} [10, Ch. V, Th. (3.5)]

Lemma 2.1. *For every $\varphi \in L^2(\rho d\mu_u)$, one has the representation*

$$\varphi(w(\tau)) = \tilde{T}\varphi(u) + \int_0^\tau \nabla \tilde{T}\varphi(w(t)) d\tilde{w}(t), \quad Q_u - a.s. \quad (2.8)$$

Subsequently, we find the Itô-Wiener expansion for the random variable $\psi(w(t))$ with respect to the Wiener process \hat{w} .

Lemma 2.2. *For every $\psi \in L^2(\mu_{t,u})$ the Itô-Wiener expansion of $\psi(w(t))$ has the form*

$$\begin{aligned} \psi(w(t)) = \sum_{n=0}^{\infty} \int_{0 < t_1 < \dots < t_n < t} \dots \int \alpha(t, u)^{-1} & \left(\alpha(t_1, \cdot) \tilde{T}_{t_1}^k \alpha(t_2 - t_1, \cdot) \nabla \tilde{T}_{t_2 - t_1}^k \dots \right. \\ & \left. \alpha(t - t_n, \cdot) \nabla \tilde{T}_{t - t_n}^k \varphi \right) (u) d\hat{w}_{t_1}(t_1) \dots d\hat{w}_{t_n}(t_n). \end{aligned} \quad (2.9)$$

The theorem 2.1 follows by substituting $\psi = \nabla \tilde{T}\varphi$ in (2.9) and inserting the right-hand side of (2.9) into (2.8).

3 Clark Representation Formula with respect to the Measure Q_u . Proof of the Lemma 2.1

Proof. 1) At first we will prove that the function $\tilde{T}\varphi$ is smooth and satisfies the equation

$$(\nabla \tilde{T}\varphi, \nabla \log \beta) + \frac{1}{2} \Delta \tilde{T}\varphi = 0 \quad (3.10)$$

in G . Indeed,

$$\tilde{T}\varphi(v) = \frac{\mathbb{E}_v 1_{\tau < \infty} \rho(w(\tau)) \varphi(w(\tau))}{\beta(v)} \quad (3.11)$$

is the ratio of two harmonic functions [9, Ch. 4, Th. 3.7] (for the numerator the condition $\varphi \in L^2(\rho d\mu_u)$ is used). The equation (3.10) is checked by straightforward calculation.

2) We will prove the relation (2.8) for bounded and continuous functions φ and ρ , the other cases being covered by the usual limiting procedure. Let $\{G_n\}_{n \geq 1}$ be a sequence of open relatively compact sets, such that $\overline{G_n} \subset G$ and $G = \bigcup_{n=1}^{\infty} G_n$. Denote τ_n be the exit time from G_n :

$$\tau_n = \inf\{t \geq 0 : w(t) \notin G_n\}.$$

The convergence $\tau_n \rightarrow \tau$, $n \rightarrow \infty$, holds.

From the relation (2.5) it follows that the stopped process $w(\cdot \wedge \tau_n)$ satisfies the SDE

$$dw(s) = \nabla \log \beta(w(s)) ds + d\tilde{w}(s), \quad 0 \leq s \leq \tau_n.$$

Applying the Itô formula to the function $\tilde{T}\varphi$ and the process $w(\cdot \wedge \tau_n)$, and using (3.10), one gets the representation

$$\tilde{T}\varphi(w(\tau_n)) = \tilde{T}\varphi(u) + \int_0^{\tau_n} \nabla \tilde{T}\varphi(w(s)) d\tilde{w}(s), \quad Q_u - a.s.$$

It remains to check that $\tilde{T}\varphi(w(\tau_n)) \rightarrow \tilde{T}\varphi(w(\tau))$. As the function φ is bounded, one has

$$\sup_{n \geq 1} \int_0^\infty \mathbb{E}_u 1_{\tau_n > s} (\nabla \tilde{T}\varphi(w(s)))^2 ds < \infty.$$

Now, the convergence $\tau_n \rightarrow \tau$, $n \rightarrow \infty$, implies the convergence

$$\int_0^{\tau_n} \nabla \tilde{T}\varphi(w(s)) d\tilde{w}(s) \xrightarrow{L^2(Q_u)} \int_0^\tau \nabla \tilde{T}\varphi(w(s)) d\tilde{w}(s), \quad n \rightarrow \infty.$$

It remains to check that $\tilde{T}\varphi(w(\tau_n)) \rightarrow \varphi(w(\tau))$, $n \rightarrow \infty$. By [9, Ch. 4, Th. 2.3] the point $w(\tau)$ is the regular point for the Dirichlet problem on G . The needed convergence follows from the representation (3.11). \square

4 The Krylov-Veretennikov Formula. Proof of the Lemma 2.2

Proof. The kernels a_n in the expansion

$$\psi(w(t)) = \sum_{n=0}^{\infty} \int_{0 < t_1 < \dots < t_n < t} \dots \int a_n(t_1, \dots, t_n) d\hat{w}_t(t_1) \dots d\hat{w}_t(t_n)$$

will be recovered from the expression

$$\begin{aligned} \mathbb{E}_{Q_{t,u}} \psi(w(t)) \int_{0 < t_1 < \dots < t_n < t} \dots \int b_n(t_1, \dots, t_n) d\hat{w}_t(t_1) \dots d\hat{w}_t(t_n) &= \int_{0 < t_1 < \dots < t_n < t} \dots \int \alpha(t, u)^{-1} \\ &\left(\alpha(t_1, \cdot) \tilde{T}_{t_1}^k \alpha(t_2 - t_1, \cdot) \nabla \tilde{T}_{t_2 - t_1}^k \dots \alpha(t - t_n, \cdot) \nabla \tilde{T}_{t - t_n}^k \varphi \right)(u) b_n(t_1, \dots, t_n) dt_1 \dots dt_n, \end{aligned}$$

in which b_n is a deterministic square integrable function. By induction, it is enough to check that for any square integrable \hat{w} -adapted process $\{g(s)\}_{0 \leq s \leq t}$, one has

$$\mathbb{E}_{Q_{t,u}} \psi(w(t)) \int_0^t g(s) d\hat{w}_t(s) = \int_0^t \frac{\alpha(s, u)}{\alpha(t, u)} \mathbb{E}_{Q_{s,u}} \alpha(t - s, w(s)) \nabla \tilde{T}_{t-s}^k \psi(w(s)) g(s) ds. \quad (4.12)$$

To do it note the equalities, which follow from (2.6) and lemma 2.1

$$\begin{aligned} \int_0^t g(s) d\hat{w}_t(s) &= \int_0^t g(s) d\tilde{w}(s) - \int_0^t g(s) \nabla \log \alpha(t - s, w(s)) ds, \quad Q_{t,u} - \text{a.s.}, \\ 1_{\tau > t} \psi(w(t)) &= T_t^k \psi(u) + \int_0^{t \wedge \tau} \nabla T_{t-s}^k \psi(w(s)) d\tilde{w}(s), \quad Q_u - \text{a.s.} \end{aligned}$$

Consequently,

$$\mathbb{E}_{Q_{t,u}} \psi(w(t)) \int_0^t g(s) d\hat{w}_t(s) =$$

$$\begin{aligned}
&= \mathbb{E}_{Q_{t,u}} \psi(w(t)) \int_0^t g(s) d\tilde{w}_t(s) - \mathbb{E}_{Q_{t,u}} \psi(w(t)) \int_0^t g(s) \nabla \log \alpha(t-s, w(s)) ds = \\
&= \alpha(t, u)^{-1} \left(\mathbb{E}_{Q_u} 1_{\tau > t} \psi(w(t)) \int_0^t g(s) d\tilde{w}_t(s) - \right. \\
&\quad \left. - \mathbb{E}_{Q_u} 1_{\tau > t} \psi(w(t)) \int_0^t g(s) \nabla \log \alpha(t-s, w(s)) ds \right) = \\
&= \alpha(t, u)^{-1} \left(\int_0^t \mathbb{E}_{Q_u} 1_{\tau > s} \nabla T_{t-s}^k \psi(w(s)) g(s) ds - \right. \\
&\quad \left. - \int_0^t \mathbb{E}_{Q_u} 1_{\tau > s} T_{t-s}^k \psi(w(s)) g(s) \nabla \log \alpha(t-s, w(s)) ds \right) = \\
&= \alpha(t, u)^{-1} \int_0^t \mathbb{E}_{Q_u} 1_{\tau > s} \left(\nabla T_{t-s}^k \psi(w(s)) - T_{t-s}^k \psi(w(s)) \nabla \log \alpha(t-s, w(s)) \right) g(s) ds = \\
&= \int_0^t \frac{\alpha(s, u)}{\alpha(t, u)} \mathbb{E}_{Q_{s,u}} \alpha(t-s, w(s)) \nabla \widetilde{T_{t-s}^k} \psi(w(s)) g(s) ds.
\end{aligned}$$

The equality (4.12) is proved. \square

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